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LETTER TO THE EDITOR

Two-dimensional quasi-exactly solvable models and classical orthogonal polynomials

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Abstract. New types of two-dimensional quasi-exactly solvable models are found. They are obtained from one-dimensional finite-difference equations with the help of the generating function which is composed from a set of functions obeying simple recurrence equations. In particular, this set can consist of some kinds of classical orthogonal polynomials. It is shown that the models found are described in terms of $SO(2,1)$ and Heisenberg–Weyl dynamical algebras.

Quasi-exactly solvable models (QESM) are characterized by the following feature: in the infinite-dimensional space of states of a quantum system there exists finite-dimensional subspace within which eigenvalues and eigenfunctions correspond to roots of a finite algebraic equation (see reviews [1]–[3]). This property is ultimately connected with the existence of hidden dynamical algebra: the Hamiltonian can be expressed in terms of linear differential operators obeying commutation relations for some Lie algebra. Finite-dimensional representation of this algebra ensures finiteness of the corresponding subspace with the algebraized part of the spectrum.

In a somewhat different approach, the starting point is the matrix structure of a finite-difference equation [4, 5] admitting finite solutions for quantities a_n:

$$(\varepsilon_2 n^2 + \varepsilon_1 n - \varepsilon_0)a_n + (n+1)[\alpha_0 + \alpha_1(n+1)]a_{n+1} + (n-N-1)[\beta + \beta_1(n-1)]a_{n-1} + \gamma(n+1)(n+2)a_{n+2} + \delta(n-N-1)(n-N-2)a_{n-2} = 0. \tag{1}$$

Here $N > 0$ is an integer, $a_n = 0$ for $n > N$ and $n < 0$.

Multiplying (1) by x^n and carrying out summation with respect to n one can obtain the closed differential equation for the generating function [4, 5]

$$\Phi = \sum_{n=0}^{N} a_n x^n \tag{2}$$

which coincides with the most general equation in the group approach based on $SU(2)$ algebra (equation (11) of [6]). Direct generalization of (2) to the two-dimensional case leads to the generating function [5]

$$\Phi = \sum_{n,m} a_{nm} x^n y^m \qquad 0 \leqslant n \leqslant N, 0 \leqslant m \leqslant M. \tag{3}$$

Two-index quantities a_{nm} obey a finite-difference equation which generalizes (1). Two-dimensional QESM obtained by this approach was discussed in detail in [5].

At the same time, the solution (3) does not represent the most general generating function for two-dimensional QESM. One can show that it corresponds only to $SU(2)\times SU(2)$ algebra [7] whereas QESM based on $SO(3)$ and $SU(3)$ algebras [8] do not belong to this type.

The aim of our paper is to present new types of two-dimensional QESM. Their generating function is essentially different from (3) and is constructed with the help of functions which, in particular cases, are reduced to some types of classical orthogonal polynomials. It turns out that QESM obtained in this way are associated with $SO(2,1)$ and Heisenberg–Weyl algebras. Thus, they represent examples of two-dimensional QESM based on infinite-dimensional representations of Lie algebras whereas in [8] only finite-dimensional ones were used.

The main idea consists of using the generating function

$$\Phi = \sum_{n=0}^{N} a_n x^n F_n(y) \tag{4}$$

where $F_n(y)$ is some set of given functions with suitable properties. Note that even if $F_n(y)$ are polynomials, (4) is not reduced to (3) since all coefficients at degrees of x and y in (4) are expressed in terms of N quantities a_n whereas in (3) there are NM coefficients a_{nm}.

As will be clear from what follows it is convenient to choose F_n obeying the recurrence relations

$$nF_{n-1} = \left(-f\frac{\partial}{\partial y} + ng + h_-\right) F_n \tag{5a}$$

and

$$(n+1)F_{n+1} = \left(f\frac{\partial}{\partial y} + ng + h_+\right) F_n \tag{5b}$$

$n \geqslant 0$, $f = f(y)$, $g = g(y)$, $h_\pm = h_\pm(y)$.

Now multiply (1) by $x^n F_n(y)$ and sum with respect to n. To demonstrate the essence of the method, let us describe in detail what happens to the term in (1) proportional to α_0:

$$\sum_{n=0}^{N} a_{n+1}(n+1)x^n F_n(y) = \sum_{n=-1}^{N}(\ldots) = \sum_{\substack{m=0\\ m=n+1}}^{N} a_m x^{m-1} m F_{m-1}$$

$$= x^{-1}\sum_{m=0}^{N} a_m x^m \left(-f\frac{\partial}{\partial y} + mg + h_-\right) F_m = -\frac{f}{x}\Phi_y + g\Phi_x + \frac{h_-\Phi}{x}. \tag{6}$$

In the second equality it was taken into account that $a_{N+1} = 0$.

All terms in (1) containing a_n, a_{n+1} and a_{n+2} are transformed in a similar way. The terms with a_{n-1} generate combinations of Φ and its derivatives only if the coefficient at a_{n-1} is proportional to n. As far as terms a_{n-2} are concerned they prevent obtaining the closed equation for Φ since N (which enters, without fail, the coefficient at a_{n-2}) creates obstacles to using (5) on which the derivation is based.

Thus, the following conditions must be satisfied for Φ to obey the closed differential equation:

$$\beta_1 = \beta \qquad \delta = 0. \tag{7}$$

The equation has the general form

$$-g^{\mu\nu}\frac{\partial^2\Phi}{\partial x^\mu \partial x^\nu} + T^\mu \frac{\partial \Phi}{\partial x^\mu} + V\Phi = 0. \tag{8}$$

The coefficients in our case equal

$$\begin{aligned}
&-g^{xx} = \gamma g^2 + \alpha_1 x g + \varepsilon_2 x^2 + \beta g x^3 \qquad -g^{yy} = \frac{\gamma f^2}{x^2} \\
&-2g^{xy} = -2\frac{\gamma g}{x^2} - \alpha_1 f + \beta x^2 f \\
&T^x = \frac{\gamma}{x}(2gh_- - fg') + \alpha_0 g + \alpha_1(g + h_-) + (\varepsilon_1 + \varepsilon_2)x + \beta x^2(h_+ + g(1-N)] \\
&T^y = \frac{\gamma f}{x^2}(2h_- - g - f') - N\beta x f - \frac{\alpha_0 f}{x} \\
&V = \frac{\alpha_0 h_-}{x} - \varepsilon_0 - \beta x N h_+ + \frac{\gamma}{x^2}(h_-^2 - h_- g - h'_- f).
\end{aligned} \tag{9}$$

Thus, the second-order differential equation is obtained for which there exists solutions (4) where a_n obey the finite-difference equation (1) and represent its finite solution, eigenvalues ε_0 being the roots of an algebraic equation. Therefore, the equation obtained belongs to the QESM type. This conclusion is valid even if functions $F_n(y)$ cannot be found in the explicit form.

Generally speaking, (8) can be reduced to the standard form of the Schrödinger equation 'Laplacian plus potential' only under additional conditions of integrability [8] which in our case must represent some relations between coefficients of (1). Finding these relations in the explicit form as well as the analysis of concrete properties of QESM (cf [5], [8]) is beyond the scope of the present paper being the subject of a separate investigation.

It is worth stressing that such important particular cases as some classical orthogonal polynomials are described by (5*a*) and (5*b*): Legendre polynomials—$f = y^2 - 1$, $g = y$, $h_+ = y$, $h_- = 0$; Laguerre polynomials L_n^0—$f = y$, $g = 1$, $h_+ = 1 - y$, $h_- = 0$.

For Laguerre polynomials $L_n^\alpha \equiv \frac{e^y y^{-\alpha}}{n!}\frac{d^n}{dy^n}(e^{-y}y^{n+\alpha})$ with $\alpha \neq 0$ it is impossible to obtain the closed differential equation for the generating function (4) in the general case. However, one manages to derive it if, in addition to (7), $\gamma = 0 = \alpha_0 = \alpha_1$. The corresponding equation has the same form (8) where now

$$\begin{aligned}
&-g^{xx} = \varepsilon_2 x^2 + \beta x^3 \qquad -g^{yy} = 0 \qquad -2g^{xy} = \beta x^2 y \\
&T^x = x(\varepsilon_1 + \varepsilon_2) + \beta x^2(\alpha + 2 - N - y) \\
&T^y = -N\beta x y \\
&V = -\varepsilon_0 + \beta x N(y - 1 - \alpha).
\end{aligned} \tag{12}$$

Apart from (5), consider now another case of relations which enables us to obtain QESM with the generating function (4). As the starting point we choose the Hermite polynomials, $F_n = H_n(y)$:

$$\begin{aligned}
&H_{n+1} = \left(2y - \frac{d}{dy}\right) H_n \\
&nH_{n-1} = \frac{1}{2}\frac{dH_n}{dy}.
\end{aligned} \tag{13}$$

The closed differential equation (8) for Φ can now be obtained in the same manner as before if

$$\beta_1 = 0 = \delta \tag{14}$$

instead of (7). The coefficients of (8) take the form

$$\begin{aligned}
-g^{xx} &= \varepsilon_2 x^2 \\
-g^{yy} &= \frac{\gamma}{4x^2} \\
-2g^{xy} &= \frac{\alpha_1}{2} - \beta x^2 \\
T^x &= (\varepsilon_1 + \varepsilon_2)x + 2\beta x^2 y \\
T^y &= \frac{\alpha_0}{2x} + \beta N x \\
V &= -\varepsilon_0 - 2N\beta x y.
\end{aligned} \tag{15}$$

Note that (5a) and (5b) can be rewritten in the form

$$\begin{aligned}
K_+\varphi_n &= (n+1)\varphi_{n+1} \\
K_-\varphi_n &= n\varphi_{n-1} \\
K_0\varphi_n &= (n+\tfrac{1}{2})\varphi_n
\end{aligned} \tag{16}$$

where $\varphi_n = x^n F_n(y)$ and operators K_i take the form

$$\begin{aligned}
K_+ &= xf\frac{\partial}{\partial y} + gx^2\frac{\partial}{\partial x} + xh_+ \\
K_- &= -x^{-1}f\frac{\partial}{\partial y} + g\frac{\partial}{\partial x} + x^{-1}h_- \\
K_0 &= x\frac{\partial}{\partial x} + k \\
k &= \tfrac{1}{2}.
\end{aligned} \tag{17}$$

It follows from (16) that operators K_i must obey the commutation relations typical of generators of $SO(2,1) \approx SU(1,1)$ group:

$$\begin{aligned}
[K_0, K_\pm] &= \pm K_\pm \\
[K_-, K_+] &= 2K_0.
\end{aligned} \tag{18}$$

Equations (16)–(18) are consistent with each other if

$$\begin{aligned}
g^2 - fg' &= 1 \\
g\rho - f\rho' &= 1 \\
h_+ + h_- &\equiv 2k\rho(y).
\end{aligned} \tag{19}$$

The invariant Casimir operator is

$$C = K_0^2 - \tfrac{1}{2}(K_-K_+ + K_+K_-). \tag{20}$$

The relations (16) realize the positive discrete series representation $\mathcal{D}^+(k)$ with the Bargmann index $k = 1/2$, φ_n serving as basis solutions.

In the subspace with a given k

$$C\varphi_n = k(k-1)\varphi_n. \tag{21}$$

For functions φ_n this is not an identity but rather the condition of compatibility of (5*a*) and (5*b*). One can verify by direct calculation that for cases (10) and (11), (21) reduces to the standard equation for Legendre and Laguerre polynomials respectively.

The approach in question can be generalized to arbitrary $k > 0$. However, it turns out that for $k \neq 1/2$, (8) has less structure than for $k = 1/2$. For this reason I will not discuss that case further.

QESM with the generating function (4) based on Hermite polynomials have hidden dynamical algebra of quite different nature. Introduce functions

$$\Psi_n = \frac{x^n H_n(y)}{\sqrt{n!}}. \tag{22}$$

Then the recurrence relations (13) for Hermite polynomials can be rewritten in the form

$$\begin{aligned} b^+\Psi_n &= \sqrt{n+1}\,\Psi_{n+1} \\ b\Psi_n &= \sqrt{n}\,\Psi_{n-1} \\ \hat{n}\Psi_n &= n\Psi_n. \end{aligned} \tag{23}$$

Here

$$\begin{aligned} b &= (2x)^{-1}\frac{\partial}{\partial y} \qquad b^+ = x\left(2y - \frac{\partial}{\partial y}\right) \\ \hat{n} &= x\frac{\partial}{\partial x}. \end{aligned} \tag{24}$$

These operators obey the commutation relations

$$\begin{aligned} [b, b^+] &= 1 \\ [\hat{n}, b^+] &= b^+ \\ [\hat{n}, b] &= -b. \end{aligned} \tag{25}$$

Although $\hat{n} \not\equiv b^+b$

$$\hat{n}\Psi_n = b^+b\Psi_n \tag{26}$$

for arbitrary Ψ_n or their linear combination entering the generating function (4), i.e. in the space of quasi-exact solutions.

By virtue of (24), (26) can be easily verified to coincide with the standard equation for Hermite polynomials.

Thus, in this case the relevant dynamical group is the Heisenberg–Weyl one, b^+ and b playing the role of creation and annihilation operators, b^+b being the operator of particle's number.

Note that examples of QESM described by the Heisenberg–Weyl algebra were indicated in [9] (the model of interacting oscillators and an oscillator interacting with spin). A general discussion about using infinite-dimensional representations of Lie algebras for QESM in a more mathematical setting is contained in [10]. In both cases [9], [10] it was assumed that a system possesses a special kind of symmetry due to an additional integral of motion. The models obtained in the present paper are free of such an assumption. They represent new examples of QESM described by infinite-dimensional representations of Lie algebras.

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